Component-based Reduced Basis for Eigenproblems

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Abstract

A component-based approach is introduced for fast and flexible solution of parameter-dependent eigenproblems. We consider a shifted version of the eigenproblem where the left hand side operator corresponds to an equilibrium between the stiffness operator and a weighted mass operator. This permits to apply the Static Condensation Reduced Basis Element method, a domain synthesis approach with reduced basis approximation at the intradomain level. We provide eigenvalue a posteriori error estimators and we present various numerical results of modal analysis of structures. We are able to obtain several orders of magnitude speed-up compared to a classical Finite Element Method.

Keywords: Eigenproblems, Domain Synthesis, Reduced Basis, *A Posteriori* Error Estimation

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1. Introduction

In structural analysis, eigenvalue computation is necessary to find the periods at which a structure will naturally resonate. This is especially important for instance in building engineering, to make sure that a building's natural frequency does not match the frequency of expected earthquakes. In the case of resonance, a building can endure large deformations and important structural damage, and possibly collapse. The same considerations apply to automobile and truck frames, where it is important to avoid resonance with the engine frequencies. Eigenproblems also appear when considering wind loads, rotating machinery, aerospace structures; in some cases it is also desirable to design a structure for resonance, like certain microelectromechanical systems.

With improvement in computer architecture and algorithmic methods, it is now possible to tackle large-scale eigenvalue problems with millions of degrees of freedom; however the computations are still heavy enough to preclude usage in a *many-query* context, such as interactive design of a parameter-dependent system. In this paper, we present an approach for fast solution of eigenproblems on large systems that present a component-based structure – such as building structures.

For the numerical solutions of partial differential equations (PDE) in component-based systems, several computational methods have been introduced to take advantage of the component-based structure. The main idea of these methods is to perform domain decomposition, and to use a common model order reduction for each family of similar components. The first and classical approach is the component

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14. ABSTRACT

A component-based approach is introduced for fast and exible solution of parameter-dependent eigenproblems. We consider a shifted version of the eigenproblem where the left hand side operator corresponds to an equilibrium between the sti ness operator and a weighted mass operator. This permits to apply the Static Condensation Reduced Basis Element method, a domain synthesis approach with reduced basis approximation at the intradomain level. We provide eigenvalue a posteriori error estimators and we present various numerical results of modal analysis of structures. We are able to obtain several orders of magnitude speed-up compared to a classical Finite Element Method.

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mode synthesis (CMS) [1, 2], which combines static condensation at the interdomain level with eigen-modal truncation at the intradomain level. For parameter-dependent equations, an alternative model order reduction is to use reduced basis (RB) approximations. This idea was first introduced in the reduced basis element (RBE) method [3], with a Mortar approach for domain decomposition. In this case the RB approximations provide a model order reduction with certified accuracy over a specified range of parameters, and the same RB spaces can be used for similar components with different parameter values.

In [4], a static condensation RBE (SCRBE) approach is developed for elliptic problems. It brings together ideas of CMS and RBE by considering standard static condensation at the interdomain level and then RB approximation at the intradomain level. In an Offline stage performed once, the RB space for a particular component is designed to reflect all possible function variations on the component interfaces (which we shall denote ports); components are thus completely interchangeable and interoperable. During the Online stage, any system can be assembled from multiple instantiations of components from a predefined library; we can then compute the system solution for different values of the component parameters in a prescribed parameter domain. The Online stage of the SCRBE is much more flexible than both the Online stage for the standard RB method, in which the system is already assembled and only parametric variations are permitted, and the Online stage of the classical (non-static-condensation) RBE method, in which the RB intradomain spaces already reflect anticipated connectivity.

In this paper, we present an extension of the SCRBE to eigenproblems. The new aspects are the following. First, the SCRBE normally takes advantage of linearity, which is lost when considering eigenproblems. Hence we begin by reformulating the eigenproblem using a shift σ of the spectrum in order to recover a linear problem. Finding the eigenvalues is then performed at a higher level: using a direct search method, we find the values of the shift σ that correspond to singular systems. Second, we provide *a posteriori* error estimators of the eigenvalues, not only with respect to RB approximations but also in the context of port reduction.

In the context of CMS approaches for eigenproblems, out method provides some important features: treatment of parameter-dependent systems (as explained above), optimal convergence, and port reduction. The classical CMS only achieves a polynomial convergence rate [5, 6] with respect to the number of eigenmodes used at the intradomain level. This can be improved to an infinite convergence rate by using overlaping components [6], but at the expense of losing simplicity and flexibility of component connections. Our method somehow provides an optimal trade-off since it retains the interface treatment of classical CMS – allowing flexibility of component connections – while achieving an exponential convergence rate with respect to the size of RB spaces at the intradomain level.

We also provide port reduction so as to increase even more the speed up. Recent CMS contributions consider several port economizations (or interface reduction strategies): an eigenmode expansion (with subsequent truncation) for the port degrees of freedom is proposed in [5, 7]; an adaptive port reduction procedure based on a posteriori error estimators for the port reduction is proposed in [8]; and an alternative port reduction approach, with a different bubble function approximation space, is proposed

for time-dependent problems in [9]. We can not directly apply CMS port reduction concepts in the parameter-dependent context, as the chosen port modes must be able to provide a good representation of the solution for any value of the parameters. In this paper, we adapt to parameter-dependent eigenproblems a port approximation and a posteriori error bound framework introduced in [10] for parameter-dependent linear elliptic problems.

The paper proceeds as follows. In Section 2, we present the general eigenproblem and its shifted formulation; we then describe the static condensation procedure. In Section 3, we add reduced basis approximations and develop *a posteriori* error estimators for the eigenvalues with respect to the corresponding values obtained by the "truth" static condensation of Section 2. In section 4, we introduce port reduction and provide as well *a posteriori* error estimators for the eigenvalues. Finally, in Section 5, we present numerical results on bridge structures to illustrate the computational efficiency of the approach.

2. Formulation

2.1. Problem statement

We suppose that we are given an open domain $\Omega \subset \mathbb{R}^d$, $d = 1, 2$ or 3, with boundary $\partial\Omega$. We then let X denote the Hilbert space

$$X \equiv \{v \in H^1(\Omega) \colon v|_{\partial\Omega_D} = 0\},$$

where $\partial\Omega_D \subset \partial\Omega$ is the portion of the boundary on which we enforce Dirichlet boundary conditions. We suppose that X is endowed with an inner product $(\cdot, \cdot)_X$ and induced norm $\|\cdot\|_X$. Recall that for any domain $\mathcal{O}$ in $\mathbb{R}^d$,

$$H^1(\mathcal{O}) \equiv \{v \in L^2(\mathcal{O}) \colon \nabla v \in (L^2(\mathcal{O}))^d\}, \text{ where } L^2(\mathcal{O}) \equiv \{v \text{ measurable over } \mathcal{O} \colon \int_{\mathcal{O}} v^2 \text{ finite}\}.$$

Furthermore, let $Y \equiv L^2(\Omega)$.

We now introduce an abstract formulation for our eigenvalue problem. For any $\mu \in \mathcal{D}$, let $a(\cdot, \cdot; \mu) : X \times X \to \mathbb{R}$, and $m(\cdot, \cdot; \mu) : X \times X \to \mathbb{R}$ denote continuous, coercive, symmetric bilinear form with respect to X and Y, respectively. We suppose that $X^{\mathcal{N}} \subset X$ is a finite element space of dimension $\mathcal{N}$. Given a parameter $\mu \in \mathcal{D} \subset \mathbb{R}^P$, where $\mathcal{D}$ is our parameter domain of dimension P, we find the set of eigenvalues and eigenvectors $(\lambda(\mu), u(\mu))$, where $\lambda(\mu) \in \mathbb{R}_{>0}$ and $u(\mu) \in X^{\mathcal{N}}$ satisfy

$$a(u(\mu), v; \mu) = \lambda(\mu) m(u(\mu), v; \mu), \quad \forall v \in X^{\mathcal{N}}, \tag{1}$$

$$m(u(\mu), u(\mu); \mu) = 1. \tag{2}$$

We assume for simplicity that the eigenvalues $\lambda_n(\mu)$ are distinct, of mutiplicity one, and sorted such that $0 < \lambda_1(\mu) < \lambda_2(\mu) \ldots < \lambda_{\mathcal{N}}(\mu)$.

We now define a surrogate eigenvalue problem that will be convenient for subsequent developments. For a given "shift factor" $\sigma \in \mathbb{R}_{\geq 0}$, we modify (1), (2) such that for any $\mu \in \mathcal{D}$, we find $\tau(\mu, \sigma) \in \mathbb{R}$ and $\chi(\mu, \sigma) \in X^{\mathcal{N}}$ that satisfy

$$\mathcal{B}(\chi(\mu, \sigma), v; \mu; \sigma) = \tau(\mu, \sigma) a(\chi(\mu, \sigma), v; \mu), \quad \forall v \in X^{\mathcal{N}}, \tag{3}$$

$$a(\chi(\mu, \sigma), \chi(\mu, \sigma); \mu) = 1. \tag{4}$$

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Here

$$\mathcal{B}(w, v; \mu; \sigma) \equiv a(w, v; \mu) - \sigma m(w, v; \mu) \tag{5}$$

is our "shifted" bilinear form. Note that we change the bilinear form on the right hand side from $m(\cdot, \cdot)$ to $a(\cdot, \cdot)$, which corresponds to a different norm. This choice is motivated by error estimation, presented later in the paper, as it permits to derive relative error estimates for the eigenvalue $\lambda_n(\mu)$.

We also sort the set of eigenvalues such that $\tau_1(\mu, \sigma) < \tau_2(\mu, \sigma) \ldots < \tau_{\mathcal{N}}(\mu, \sigma)$ – note that due to the shift the first eigenvalues can now be negative. It is also clear that $\chi_n(\mu, \sigma) = \frac{1}{\sqrt{\lambda_n(\mu)}} u_n(\mu)$ for any $\sigma \in \mathbb{R}$, so we shall henceforth write $\chi_n(\mu)$. Also

$$\tau_n(\mu, \sigma) = \frac{\lambda_n(\mu) - \sigma}{\lambda_n(\mu)}, \tag{6}$$

so that

$$\tau_n(\mu, \sigma) > 0, \qquad \text{if } 0 \le \sigma < \lambda_n(\mu), \tag{7}$$

$$\tau_n(\mu, \sigma) = 0, \qquad \text{if } \sigma = \lambda_n(\mu), \tag{8}$$

$$\tau_n(\mu, \sigma) < 0, \qquad \text{if } \sigma > \lambda_n(\mu), \tag{9}$$

for $n = 1, \ldots, \mathcal{N}$.

2.2. Static Condensation

We now move to the component level. We suppose that the system domain is naturally decomposable into I interconnected parametrized components. Each component i is associated with a subdomain Ω_i, where

$$\overline{\Omega} = \bigcup_{i=1}^{I} \overline{\Omega}_i, \qquad \Omega_i \cap \Omega_{i'} = \emptyset, \text{ for } i \ne i' .$$

We say that components i and i' are connected at global port p if $\overline{\Omega}_i \cap \overline{\Omega}_{i'} = \Gamma_p \ne \emptyset$. We also say that $\gamma_i^j = \Gamma_p$ and $\gamma_{i'}^{j'} = \Gamma_p$ are local ports of components i and i' respectively. We denote by n^Γ the total number of global ports in the system, and we denote by n_i^γ the total number of local ports in component i. We assume that the FE space $X^{\mathcal{N}}$ conforms to our components and ports, hence we can define the discrete spaces $X_i^{\mathcal{N}}$ and $Z_p^{\mathcal{N}}$ that are simply the restrictions of $X^{\mathcal{N}}$ to component i and global port p. For given i, let $X_{i;0}^{\mathcal{N}}$ denote the "component bubble space" — the restriction of $X^{\mathcal{N}}$ to Ω_i with homogeneous Dirichlet boundary conditions on each $\gamma_i^j, 1 \le j \le n_i^\gamma$,

$$X_{i;0}^{\mathcal{N}} \equiv \{v|_{\Omega_i} : v \in X^{\mathcal{N}}; \ v|_{\gamma_i^j} = 0, \ 1 \le j \le n_i^\gamma\}.$$

We denote by $\mathcal{N}_p^\Gamma$ the dimension of the port space $Z_p^{\mathcal{N}}$ associated with global port p, and we say that the global port p has $\mathcal{N}_p^\Gamma$ degrees of freedom (dof). For each component i, we denote by k' a local port dof number, and K_i the total numbers of dof on its local ports, such that $1 \le k' \le K_i$. We then introduce the map $\mathcal{P}_i(k') = (p, k)$ which associate a local port dof k' in component i to its global port representation: global port p and dof k, $1 \le k \le \mathcal{N}_p^\Gamma$.

To formulate our static condensation procedure we must first introduce the basis functions for the port space $Z_p^{\mathcal{N}}$ as $\{\zeta_{p,1}, \cdots, \zeta_{p,\mathcal{N}_p^\Gamma}\}$. The particular choice for these functions is not important for now,

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but it becomes critical when dealing with port reduction – we refer to Section 4. For a local port dof number k' such that $\mathcal{P}_i(k') = (p, k)$, we then introduce the interface function $\psi_{k'}^i \in X_i^{\mathcal{N}}$, which is the harmonic extension of the associated port space basis function into the interior of the component domain Ω_i, and satisfies

$$\int_{\Omega_i} \nabla \psi_{k'}^i \cdot \nabla v = 0, \ \ \forall v \in X_{i;0}^{\mathcal{N}}, \tag{10}$$

$$\psi_{k'}^i = \begin{cases} \zeta_{p,k} & \text{on} \ \ \Gamma_p \\ 0 & \text{on} \ \gamma_i^j \neq \Gamma_p, \ 1 \leq j \leq n_i^\gamma. \end{cases} \tag{11}$$

If components i and j are connected, then for each matching local port dofs k_i and k_j such that $\mathcal{P}_i(k_i) = \mathcal{P}_j(k_j) = (p, k)$, we define the global interface function $\Psi_{p,k} \in X^{\mathcal{N}}$ as

$$\Psi_{p,k} = \begin{cases} \psi_{k_i}^i & \text{on} \ \Omega_i \\ \psi_{k_j}^j & \text{on} \ \Omega_j \\ 0 & \text{elsewhere} . \end{cases} \tag{12}$$

We will now develop an expression for $\chi_n(\mu)$ which just involves dof on the ports by virtue of elimination of the interior dof given that $\sigma = \lambda_n(\mu)$ – starting from (13) to finally arrive at (18). Let us suppose that we set $\sigma = \lambda_n(\mu)$ (for some n) so that the right-hand side of (3) vanishes. Then, for $\chi_n(\mu) \in X^{\mathcal{N}}$ we have

$$\mathcal{B}(\chi_n(\mu), v; \mu; \sigma) = 0, \quad \text{for all } v \in X^{\mathcal{N}}.$$

We then express $\chi_n(\mu) \in X^{\mathcal{N}}$ in terms of "interface" and "bubble" contributions,

$$\chi_n(\mu) = \sum_{i=1}^I b_i(\mu, \sigma) + \sum_{p=1}^{n^\Gamma} \sum_{k=1}^{\mathcal{N}_p^\Gamma} U_{p,k}(\mu, \sigma) \Psi_{p,k}, \tag{13}$$

where the $U_{p,k}(\mu, \sigma)$ are interface function coefficients corresponding to the port p, and $b_i(\mu, \sigma) \in X_{i;0}^{\mathcal{N}}$. Here χ_n is independent of σ, but we shall see shortly that we will need b_i and $U_{k,p}$ to be σ-dependent in general.

We then restrict to a single component i to obtain

$$\mathcal{B}_i(\chi_n(\mu), v; \mu; \sigma) = 0, \quad \text{for all } v \in X_{i;0}^{\mathcal{N}}, \tag{14}$$

where $\mathcal{B}_i(w, v; \mu; \sigma) \equiv a_i(w, v; \mu) - \sigma m_i(w, v; \mu)$, and where a_i and m_i indicate the restrictions of a and m to Ω_i, respectively. Substitution of (13) into (14) leads to

$$\mathcal{B}_i(b_i(\mu, \sigma), v; \mu; \sigma) + \sum_{k=1}^{K_i} U_{\mathcal{P}_i(k)}(\mu, \sigma) \mathcal{B}_i(\psi_{i,k}, v; \mu; \sigma) = 0, \tag{15}$$

for all $v \in X_{i;0}^{\mathcal{N}}$.

It can be shown from linearity of the above equation that we can reconstruct $b_i(\mu, \sigma)$ as

$$b_i(\mu, \sigma) = \sum_{k=1}^{K_i} U_{\mathcal{P}_i(k)}(\mu, \sigma) b_{i,k}(\mu, \sigma),$$

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where $b_{i,k}(\mu, \sigma) \in X_{i;0}^{\mathcal{N}}$ satisfies

$$\mathcal{B}_i(b_{i,k}(\mu, \sigma), v; \mu; \sigma) = -\mathcal{B}_i(\psi_{i,k}, v; \mu; \sigma), \quad \forall v \in X_{i;0}^{\mathcal{N}}. \tag{16}$$

Let $(\lambda_{i,n}(\mu), \chi_{i,n}(\mu)) \in \mathbb{R} \times X_{i;0}^{\mathcal{N}}$ denote an eigenpair associated with the n local eigenproblem

$$a_i(\chi_{i,n}(\mu), v; \mu) = \lambda_{i,n}(\mu) m_i(\chi_{i,n}(\mu), v; \mu), \quad \forall v \in X_{i;0}^{\mathcal{N}}, \tag{17}$$

then, since

$$
\begin{aligned}
\inf_{v \in X_{i;0}^{\mathcal{N}}} \frac{\mathcal{B}_i(v, v; \mu; \sigma)}{\|v\|_{X,i}^2} &= \inf_{v \in X_{i;0}^{\mathcal{N}}} \frac{a_i(v, v; \mu) - \sigma m_i(v, v; \mu)}{\|v\|_{X,i}^2} \\
&\geq \inf_{v \in X_{i;0}^{\mathcal{N}}} \frac{a_i(v, v; \mu) - \sigma m_i(v, v; \mu)}{m_i(v, v; \mu)} \inf_{v \in X_{i;0}^{\mathcal{N}}} \frac{m_i(v, v; \mu)}{\|v\|_{X,i}^2} \\
&= (\lambda_{i,1}(\mu) - \sigma) \inf_{v \in X_{i;0}^{\mathcal{N}}} \frac{m_i(v, v; \mu)}{\|v\|_{X,i}^2},
\end{aligned}
$$

the bilinear form $\mathcal{B}_i(\cdot, \cdot; \mu; \sigma)$ is coercive on $X_{i;0}^{\mathcal{N}}$ if $\sigma < \lambda_{i,1}(\mu)$, where $\lambda_{i,1}(\mu)$ is the smallest eigenvalue of (17). Hence (16) has a unique solution under this condition. Note that we expect that $\lambda_{i,1}(\mu) > \lambda_1(\mu)$, and even $\lambda_{i,1}(\mu) > \lambda_n(\mu)$ for $n = 2$ or 3 or 4 — of course in practice the balance between λ_n and $\lambda_{i,n'}$ will depend on the details of a particular problem.

Now for $1 \leq k \leq \mathcal{N}_p^{\Gamma}$ and each p, let

$$\Phi_{p,k}(\mu, \sigma) = \Psi_{p,k} + \sum_{i,k' \, s.t. \, P_i(k') = (p,k)} b_{i,k'}(\mu, \sigma),$$

and let us define the "skeleton" space $X_{\mathcal{S}}(\mu, \sigma)$ as

$$X_{\mathcal{S}}(\mu, \sigma) \equiv \text{span}\{\Phi_{p,k}(\mu, \sigma) : 1 \leq p \leq n^{\Gamma}, 1 \leq k \leq \mathcal{N}_p^{\Gamma}\}.$$

This space is of dimension $n_{\text{sc}} = \sum_{p=1}^{n^{\Gamma}} \mathcal{N}_p^{\Gamma}$. We restrict (13) to a single component i to see that for $\sigma = \lambda_n(\mu)$ we obtain

$$\chi_n(\mu)|_{\Omega_i} = \sum_{k=1}^{K_i} U_{\mathcal{P}_i(k)}(\mu, \sigma) \left(b_{i,k}(\mu, \sigma) + \psi_{i,k} \right).$$

This then implies

$$\chi_n(\mu) = \sum_{p=1}^{n^{\Gamma}} \sum_{k=1}^{\mathcal{N}_p^{\Gamma}} U_{p,k}(\mu, \sigma) \, \Phi_{p,k}(\mu, \sigma) \in X_{\mathcal{S}}(\mu, \sigma). \tag{18}$$

Then, for $\sigma = \lambda_n(\mu)$ and $\mu \in \mathcal{D}$, we are able to solve for the coefficients $U_{p,k}(\mu, \sigma)$ from the static condensation eigenvalue problem on $X_{\mathcal{S}}(\mu, \sigma)$: find $\chi_n(\mu) \in X_{\mathcal{S}}(\mu, \sigma)$, such that

$$\mathcal{B}(\chi_n(\mu), v; \mu; \sigma) = 0, \quad \forall v \in X_{\mathcal{S}}(\mu, \sigma), \tag{19}$$

$$a(\chi_n(\mu), \chi_n(\mu); \mu) = 1. \tag{20}$$

We now relax the condition $\sigma = \lambda_n(\mu)$ to obtain the following problem: For $\sigma \in [0, \sigma_{\max}]$ and $\mu \in \mathcal{D}$, find $(\bar{\tau}_n(\mu, \sigma), \bar{\chi}_n(\mu, \sigma)) \in (\mathbb{R}, X_{\mathcal{S}}(\mu, \sigma))$, such that

$$\mathcal{B}(\bar{\chi}_n(\mu, \sigma), v; \mu; \sigma) = \bar{\tau}_n(\mu, \sigma) a(\bar{\chi}_n(\mu; \sigma), v; \mu), \quad \forall v \in X_{\mathcal{S}}(\mu, \sigma), \tag{21}$$

$$a(\bar{\chi}_n(\mu, \sigma), \bar{\chi}_n(\mu, \sigma); \mu) = 1. \tag{22}$$

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It is important to note that this new eigenproblem (21) (22) differs from (3) (4) in two ways: first, we consider a subspace $X_{\mathcal{S}}(\mu, \sigma)$ of $X^{\mathcal{N}}$, and as a consequence $\overline{\tau}_n(\mu, \sigma) \geq \tau_n(\mu, \sigma)$; second, the subspace $X_{\mathcal{S}}(\mu, \sigma)$, unlike $X^{\mathcal{N}}$, depends on σ, and furthermore only for $\sigma = \lambda_n$ does the subspace $X_{\mathcal{S}}(\mu, \sigma)$ reproduce the eigenfunction $\chi_n(\mu)$. We now show

Proposition 2.1. *Suppose $\sigma < \lambda_{i,1}(\mu)$ for each $1 \leq i \leq I$ to ensure that the static condensation is well-posed.*

(i) $\overline{\tau}_n(\mu, \sigma) \geq \tau_n(\mu, \sigma)$, $n = 1, \ldots, \dim(X_{\mathcal{S}}(\mu, \sigma))$,

(ii) $\tau_n(\mu, \sigma) = 0$ if and only if $\sigma = \lambda_n(\mu)$,

(iii) $\sigma = \lambda_n(\mu)$ if and only if there exists some $n' \leq n$ such that $\overline{\tau}_{n'}(\mu, \sigma) = 0$.

Proof. (i) The case $n = 1$ follows from the Rayleigh quotients

$$\tau_1(\mu, \sigma) = \inf_{w \in X^{\mathcal{N}}} \frac{\mathcal{B}(w, w; \mu; \sigma)}{a(w, w; \mu)}, \tag{23}$$

and

$$\overline{\tau}_1(\mu, \sigma) = \inf_{w \in X_{\mathcal{S}}(\mu, \sigma)} \frac{\mathcal{B}(w, w; \mu; \sigma)}{a(w, w; \mu)}, \tag{24}$$

and fact that $X_{\mathcal{S}}(\mu, \sigma) \subset X^{\mathcal{N}}$.

For $n > 1$, the Courant-Fischer-Weyl min-max principle [11] states that for an arbitrary n-dimensional subspace of $X^{\mathcal{N}}$, S_n, we have

$$\eta_n(\mu, \sigma) \equiv \max_{w \in S_n} \frac{\mathcal{B}(w, w; \mu; \sigma)}{a(w, w; \mu)} \geq \tau_n(\mu, \sigma). \tag{25}$$

Let $S_n \equiv \text{span}\{\overline{\chi}_m(\mu, \sigma), m = 1, \ldots, n\} \subset X_{\mathcal{S}}(\mu, \sigma)$. Then $\eta_n(\mu, \sigma) = \overline{\tau}_n(\mu, \sigma)$, and the result follows.

(ii) This equivalence is due to (8).

(iii) ($\Leftarrow$) Suppose $\sigma = \lambda_n(\mu)$ for some n, then by construction $\chi_n(\mu, \sigma) \in X_{\mathcal{S}}(\mu, \sigma)$. Since the same operator $\mathcal{B}$ appears in both (19) and (21), it follows that $\chi_n(\mu, \sigma)$ is also eigenmode for (21), (22) with corresponding eigenvalue 0. That is, for some n', $\overline{\tau}_{n'}(\mu, \sigma) = 0$ is an eigenvalue of (21),(22). Moreover $\overline{\tau}_{n+1}(\mu, \sigma) \geq \tau_{n+1}(\mu, \sigma) > 0$ so $n' \leq n$. We note that if $n > 1$, we do not necessarily capture $\chi_m(\mu)$ in $X_{\mathcal{S}}(\mu, \sigma)$ for $m = 1, 2, \ldots, n-1$, hence it is possible that $n' < n$. On the other hand, if $n = 1$, then we must have $\overline{\chi}_1(\mu, \sigma) = \chi_1(\mu, \sigma)$.

($\Rightarrow$) Suppose $\overline{\tau}_{n'}(\mu, \sigma) = 0$ for some index n'. Then $\overline{\chi}_{n'}(\mu, \sigma)$ satisfies (19), (20), or equivalently, (3), (4) for $\tau_n(\mu, \sigma) = 0$. From part (ii) of this Proposition, this implies that $\sigma = \lambda_n(\mu)$. Moreover $\overline{\tau}_{n+1}(\mu, \sigma) \geq \tau_{n+1}(\mu, \sigma) > 0$ so $n' \leq n$.

$\square$

A stronger result that will be important for the following can be obtained assuming

Hypothesis 2.1. *For $1 \leq n \leq n_{sc}$ the functions $\sigma \to \overline{\tau}_n(\mu, \sigma)$ have exactly one root.*

Corollary 2.1. *If Hypothesis 2.1 holds, $\sigma = \lambda_n(\mu)$ if and only if $\overline{\tau}_n(\mu, \sigma) = 0$.*

Proof. From Proposition 2.1(iii), we have that $\sigma = \lambda_1(\mu)$ if and only if $\overline{\tau}_1(\mu, \sigma) = 0$. Then using Hypothesis 2.1 and Proposition 2.1(iii), it is easy to show by recurrence that $\sigma = \lambda_n(\mu)$ if and only if $\overline{\tau}_n(\mu, \sigma) = 0$ for $1 \leq n \leq n_{\mathrm{sc}}$. $\qquad\square$

Lemma 2.1. *We have that*

$$\frac{\partial \tau_n(\mu, \sigma)}{\partial \sigma} = -\frac{1}{\lambda_n(\mu)}, \tag{26}$$

for each $n = 1, 2, \ldots, \mathcal{N}$.

Proof. We set $v = \chi_n(\mu)$ in (3), to obtain

$$\mathcal{B}(\chi_n(\mu), \chi_n(\mu); \mu; \sigma) = \tau_n(\mu, \sigma) a(\chi_n(\mu), \chi_n(\mu); \mu) = \tau_n(\mu, \sigma), \tag{27}$$

where we employed (4) in the last equality above. We note that $\chi_n(\mu)$ is independent of σ, and differentiate with respect to σ to obtain

$$
\begin{aligned}
\frac{\partial \tau_n(\mu, \sigma)}{\partial \sigma} &= \frac{\partial}{\partial \sigma} \mathcal{B}(\chi_n(\mu), \chi_n(\mu); \mu; \sigma) \\
&= \frac{\partial}{\partial \sigma} \left[a(\chi_n(\mu), \chi_n(\mu); \mu) - \sigma m(\chi_n(\mu), \chi_n(\mu); \mu) \right] \\
&= -m(\chi_n(\mu), \chi_n(\mu); \mu) \\
&= -\frac{1}{\lambda_n(\mu)} m(u_n(\mu), u_n(\mu); \mu) = -\frac{1}{\lambda_n(\mu)}.
\end{aligned}
$$

$\qquad\square$

However, the result does not apply to $\overline{\tau}_n(\mu, \sigma)$: we cannot apply the argument from the proposition to (21), (22) since in general $\overline{\chi}_n(\mu, \sigma)$ depends on σ. We can still state the following

Proposition 2.2. *Assuming that $\overline{\tau}_n(\mu, \cdot)$ is differentiable at $\lambda_n(\mu)$ and Hypothesis 2.1 holds, then*

$$\frac{\partial \overline{\tau}_n(\mu, \lambda_n)}{\partial \sigma} = -\frac{1}{\lambda_n(\mu)}.$$

Proof. We know that $\overline{\tau}_n(\mu, \lambda_n(\mu)) = \tau_n(\mu, \lambda_n(\mu)) = 0$, $\frac{\partial \tau_n(\mu, \sigma)}{\partial \sigma} = -\frac{1}{\lambda_n(\mu)}$ and $\overline{\tau}_n(\mu, \sigma) \geq \tau_n(\mu, \sigma)$. So we have

$$\forall h < 0, \quad \frac{\overline{\tau}_n(\mu, \lambda_n(\mu) + h)}{h} \leq -\frac{1}{\lambda_n(\mu)},$$

$$\forall h > 0, \quad \frac{\overline{\tau}_n(\mu, \lambda_n(\mu) + h)}{h} \geq -\frac{1}{\lambda_n(\mu)},$$

Since $\overline{\tau}_n(\mu, \cdot)$ is differentiable at $\lambda_n(\mu)$, we have

$$\frac{\partial \overline{\tau}_n(\mu, \lambda_n(\mu))}{\partial \sigma} = \lim_{h \to 0^-} \frac{\overline{\tau}_n(\mu, \lambda_n(\mu) + h)}{h} = \lim_{h \to 0^+} \frac{\overline{\tau}_n(\mu, \lambda_n(\mu) + h)}{h} = -\frac{1}{\lambda_n(\mu)}.$$

$\qquad\square$

To assemble an algebraic system for the static condensation eigenproblem, we insert (18) into (21), (22) to arrive at

$$\sum_{p=1}^{n^\Gamma} \sum_{k=1}^{\mathcal{N}_p^\Gamma} U_{p,k}(\mu,\sigma) \mathcal{B}(\Phi_{p,k}(\mu,\sigma), v; \mu; \sigma)$$

$$= \overline{\tau}(\mu,\sigma) \sum_{p=1}^{n^\Gamma} \sum_{k=1}^{\mathcal{N}_p^\Gamma} U_{p,k}(\mu,\sigma) a(\Phi_{p,k}(\mu,\sigma), v; \mu; \sigma), \quad \forall v \in X_{\mathcal{S}}, \tag{28}$$

$$\sum_{p=1}^{n^\Gamma} \sum_{k=1}^{\mathcal{N}_p^\Gamma} U_{p,k}(\mu,\sigma) a(\Phi_{p,k}(\mu,\sigma), \Phi_{p,k}(\mu,\sigma); \mu; \sigma) = 1. \tag{29}$$

We now define our local stiffness and mass matrices $\mathbb{A}^i(\mu,\sigma), \mathbb{M}^i(\mu,\sigma) \in \mathbb{R}^{K_i \times K_i}$ for component i, which have entries

$$
\begin{aligned}
\mathbb{A}^i_{k',k}(\mu,\sigma) &= a_i(\psi_{i,k} + b_{i,k}(\mu,\sigma), \psi_{i,k'} + b_{i,k'}(\mu,\sigma); \mu), \\
\mathbb{M}^i_{k',k}(\mu,\sigma) &= m_i(\psi_{i,k} + b_{i,k}(\mu,\sigma), \psi_{i,k'} + b_{i,k'}(\mu,\sigma); \mu),
\end{aligned}
$$

for $1 \leq k, k' \leq K_i$. We may then assemble the global system with matrices $\mathbb{B}(\mu,\sigma), \mathbb{A}(\mu,\sigma) \in \mathbb{R}^{n_{sc} \times n_{sc}}$, of dimension $n_{sc} = \sum_{p=1}^{n^\Gamma} \mathcal{N}_p^\Gamma$: given a $\sigma \in \mathbb{R}$ and $\mu \in \mathcal{D}$, we consider the eigenproblem

$$
\begin{aligned}
\mathbb{B}(\mu,\sigma)\mathbb{V}(\mu,\sigma) &= \overline{\tau}(\mu,\sigma)\mathbb{A}(\mu,\sigma)\mathbb{V}(\mu,\sigma), \tag{30}\\
\mathbb{V}(\mu,\sigma)^T \mathbb{A}(\mu,\sigma)\mathbb{V}(\mu,\sigma) &= 1, \tag{31}
\end{aligned}
$$

where

$$\mathbb{B}(\mu,\sigma) \equiv \mathbb{A}(\mu,\sigma) - \sigma\mathbb{M}(\mu,\sigma). \tag{32}$$

Assuming (2.1) holds, we can recover $\lambda_n(\mu)$ as the value of σ for which the nth eigenvalue of (30) is zero. In practice we use Brent's method [12], which is a combination of different direct search methods for root finding (bisection, secant and inverse quadratic interpolation).

3. Reduced Basis Static Condensation System

3.1. Reduced Basis Bubble Approximation

In the static condensation reduced basis element (SCRBE) method [4], we replace the FE bubble functions $b_{i,k}(\mu,\sigma)$ with reduced basis approximations. These RB approximations are significantly less expensive to evaluate (following an RB offline preprocessing step) than the original FE quantities, and hence the computational cost associated with the formation of the (now approximate) static condensation system is significantly reduced. We thus introduce the RB bubble function approximations

$$\tilde{b}_{i,k}(\mu,\sigma) \approx b_{i,k}(\mu,\sigma) \tag{33}$$

for a parameter domain $(\mu,\sigma) \in \mathcal{D} \times [0, \sigma_{\max}]$, where

$$\sigma_{\max} = \epsilon_\sigma \min_{\mu \in \mathcal{D}} \min_{1 \leq i \leq I} \lambda_{i,1}(\mu).$$

Here $\epsilon_\sigma(<1)$ is a "safety factor" which ensures that we honor the condition $\sigma < \lambda_{i,1}(\mu)$ for all $1 \le i \le I$. Next, we let

$$\widetilde{\Phi}_{p,k}(\mu,\sigma) = \Psi_{p,k} + \sum_{i,k_i \, s.t. \, P_i(k_i)=(p,k)} \tilde{b}_{i,k_i}(\mu,\sigma),$$

and define our RB static condensation space $\widetilde{X}_{\mathcal{S}}(\mu,\sigma) \subset X^{\mathcal{N}}$ as

$$\widetilde{X}_{\mathcal{S}}(\mu,\sigma) = \text{span}\{\widetilde{\Phi}_{p,k}(\mu,\sigma) : 1 \le p \le n^\Gamma, 1 \le k \le \mathcal{N}_p^\Gamma\}.$$

(Note that $\widetilde{X}_{\mathcal{S}}(\mu,\sigma) \not\subset X_{\mathcal{S}}(\mu,\sigma)$.) We then define the RB eigenproblem: given $(\mu,\sigma) \in \mathcal{D} \times [0,\sigma_{\max}]$, find the eigenpairs $(\widetilde{\overline{\tau}}_n(\mu,\sigma), \widetilde{\mathbb{V}}_n(\mu,\sigma))$ that satisfy

$$\widetilde{\mathbb{B}}(\mu,\sigma)\widetilde{\mathbb{V}}(\mu,\sigma) = \widetilde{\overline{\tau}}(\mu,\sigma)\widetilde{\mathbb{A}}(\mu,\sigma)\widetilde{\mathbb{V}}(\mu,\sigma), \tag{34}$$

$$\widetilde{\mathbb{V}}(\mu,\sigma)^T \widetilde{\mathbb{A}}(\mu,\sigma)\widetilde{\mathbb{V}}(\mu,\sigma) = 1, \tag{35}$$

where $\widetilde{\mathbb{B}}(\mu,\sigma), \widetilde{\mathbb{A}}(\mu,\sigma)$ are constructed component-by-component from

$$\widetilde{\mathbb{A}}_{k',k}^i(\mu,\sigma) = a_i(\psi_{i,k} + \tilde{b}_{i,k}(\mu,\sigma), \psi_{i,k'} + \tilde{b}_{i,k'}(\mu,\sigma); \mu),$$

$$\widetilde{\mathbb{M}}_{k',k}^i(\mu,\sigma) = m_i(\psi_{i,k} + \tilde{b}_{i,k}(\mu,\sigma), \psi_{i,k'} + \tilde{b}_{i,k'}(\mu,\sigma); \mu),$$

for $1 \le k, k' \le K_i$, and where

$$\widetilde{\mathbb{B}}^i(\mu,\sigma) \equiv \widetilde{\mathbb{A}}^i(\mu,\sigma) - \sigma\widetilde{\mathbb{M}}^i(\mu,\sigma). \tag{36}$$

3.2. Reduced Basis Error Estimator

We now consider error estimation for our RB approximations. First, since $\widetilde{X}_{\mathcal{S}}(\mu,\sigma) \subset X^{\mathcal{N}}$, by the same argument as part (i) of Proposition 2.1, we have

Corollary 3.1.

$$\widetilde{\overline{\tau}}_n(\mu,\sigma) \ge \tau_n(\mu,\sigma), \quad n = 1, 2, \ldots, n_{\text{sc}}. \tag{37}$$

$\square$

We define the residual $r_{i,k}(\cdot; \mu,\sigma) : X_{i;0}^{\mathcal{N}} \to \mathbb{R}$ for $1 \le k \le K_i$, and $1 \le i \le I$ as

$$r_{i,k}(v; \mu,\sigma) = -\mathcal{B}_i(\psi_{i,k} + \tilde{b}_{i,k}(\mu,\sigma), v; \mu,\sigma), \quad \forall v \in X_{i;0}^{\mathcal{N}},$$

and the error bound [13]

$$\|b_{i,k}(\mu,\sigma) - \tilde{b}_{i,k}(\mu,\sigma)\|_{X,i} \le \widetilde{\Delta}_{i,k}(\mu,\sigma) = \frac{\mathcal{R}_{i,k}(\mu,\sigma)}{\alpha_i^{\text{LB}}(\mu,\sigma)},$$

where

$$\mathcal{R}_{i,k}(\mu,\sigma) = \sup_{v \in X_{i;0}^{\mathcal{N}}} \frac{r_{i,k}(v; \mu,\sigma)}{\|v\|_{X,i}}$$

is the dual norm of the residual, and $\alpha_i^{\text{LB}}(\mu,\sigma)$ is a lower bound for the coercivity constant

$$\alpha_i(\mu,\sigma) = \inf_{w \in X_{i;0}^{\mathcal{N}}} \frac{\mathcal{B}_i(w, w; \mu,\sigma)}{\|w\|_{X,i}^2}. \tag{38}$$

Such a lower bound can be derived by hand for simple parametric dependences, otherwise it can be computed using a successive constraint linear optimization method [14]. It is also possible to compute during the Offline stage different coercivity constants $\alpha_i^k(\mu_k, \sigma_k)^1$ for given sampling parameters (μ_k, σ_k), and then a lower bound over the full parameter range is obtained by taking the minimum over the sampling parameters $\alpha_i^{\mathrm{LB}} = \min_k \alpha_i^k(\mu_k, \sigma_k)$. This is the latter option that is used in the results section of this paper.

We now assume that Hypothesis 2.1 holds. Suppose we have found σ_n, the nth "shift" such that $\widetilde{\mathbb{B}}(\mu, \sigma_n)$ has a zero eigenvalue, i.e. we have $\widetilde{\overline{\tau}}_n(\mu, \sigma_n) = 0$. Then our RB-based approximation to the nth eigenvalue is $\tilde{\lambda}_n(\mu) = \sigma_n$. We will now develop a first order error estimator for $\overline{\tau}_n(\mu, \sigma_n)$. We have

$$\mathbb{B}(\mu, \sigma_n)\mathbb{V}(\mu, \sigma_n) = \overline{\tau}_n(\mu, \sigma_n)\mathbb{A}(\mu, \sigma_n)\mathbb{V}(\mu, \sigma_n),$$

and hence with $\mathbb{B}(\mu, \sigma_n) \equiv \widetilde{\mathbb{B}}(\mu, \sigma_n) + \delta\mathbb{B}(\mu, \sigma_n)$, $\mathbb{A}(\mu, \sigma_n) \equiv \widetilde{\mathbb{A}}(\mu, \sigma_n) + \delta\mathbb{A}(\mu, \sigma_n)$, $\mathbb{V}(\mu, \sigma_n) \equiv \widetilde{\mathbb{V}}(\mu, \sigma_n) + \delta\mathbb{V}(\mu, \sigma_n)$, we obtain

$$(\widetilde{\mathbb{B}}(\mu, \sigma_n) + \delta\mathbb{B}(\mu, \sigma_n))(\widetilde{\mathbb{V}}(\mu, \sigma_n) + \delta\mathbb{V}(\mu, \sigma_n)) = \overline{\tau}_n(\mu, \sigma_n)(\widetilde{\mathbb{A}}(\mu, \sigma_n) + \delta\mathbb{A}(\mu, \sigma_n))(\widetilde{\mathbb{V}}(\mu, \sigma_n) + \delta\mathbb{V}(\mu, \sigma_n)). \tag{39}$$

Expansion of the above expression yields

$$\widetilde{\mathbb{B}}(\mu, \sigma_n)\delta\mathbb{V}(\mu, \sigma_n) + \delta\mathbb{B}(\mu, \sigma_n)\widetilde{\mathbb{V}}(\mu, \sigma_n) + \delta\mathbb{B}(\mu, \sigma_n)\delta\mathbb{V}(\mu, \sigma_n) =$$
$$\overline{\tau}_n(\mu, \sigma_n)(\widetilde{\mathbb{A}}(\mu, \sigma_n)\widetilde{\mathbb{V}}(\mu, \sigma_n) + \widetilde{\mathbb{A}}(\mu, \sigma_n)\delta\mathbb{V}(\mu, \sigma_n) + \delta\mathbb{A}(\mu, \sigma_n)\widetilde{\mathbb{V}}(\mu, \sigma_n) + \delta\mathbb{A}(\mu, \sigma_n)\delta\mathbb{V}(\mu, \sigma_n)), \tag{40}$$

where the identity $\widetilde{\mathbb{B}}(\mu, \sigma_n)\widetilde{\mathbb{V}}(\mu, \sigma_n) = 0$ has been employed. We then multiply through by $\widetilde{\mathbb{V}}(\mu, \sigma_n)^T$ and note that $\widetilde{\mathbb{V}}(\mu, \sigma_n)^T\widetilde{\mathbb{B}}(\mu, \sigma_n)\delta\mathbb{V}(\mu, \sigma_n) = \delta\mathbb{V}(\mu, \sigma_n)^T\widetilde{\mathbb{B}}(\mu, \sigma_n)\widetilde{\mathbb{V}}(\mu, \sigma_n) = 0$, $\widetilde{\mathbb{V}}(\mu, \sigma_n)^T\widetilde{\mathbb{A}}(\mu, \sigma_n)\widetilde{\mathbb{V}}(\mu, \sigma_n) = 1$ and neglect higher order terms to obtain

$$\overline{\tau}_n(\mu, \sigma_n) \approx \widetilde{\mathbb{V}}(\mu, \sigma_n)^T\delta\mathbb{B}(\mu, \sigma_n)\widetilde{\mathbb{V}}(\mu, \sigma_n). \tag{41}$$

We then have the following bound

$$|\widetilde{\mathbb{V}}(\mu, \sigma_n)^T\delta\mathbb{B}(\mu, \sigma_n)\widetilde{\mathbb{V}}(\mu, \sigma_n)| \leq \sum_{i=1}^I\sum_{k=1}^{K_i}\sum_{j=1}^I\sum_{l=1}^{K_j}|\widetilde{\mathbb{V}}_{\mathcal{P}_i(k)}(\mu, \sigma_n)|\widetilde{\Delta}_{i,k}(\mu, \sigma_n)\widetilde{\Delta}_{j,l}(\mu, \sigma_n)|\widetilde{\mathbb{V}}_{\mathcal{P}_j(l)}(\mu, \sigma_n)| \equiv \widetilde{\Delta}(\mu, \sigma_n). \tag{42}$$

From Proposition 2.1 part (iii), we can only infer eigenvalues of (1),(2) when $\overline{\tau}_n(\mu, \sigma) = 0$, hence (42) does not give us a direct bound on the error of $\tilde{\lambda}_n(\mu)$. However, with the assumption that $\widetilde{\Delta}(\mu, \sigma_n) \to 0$ in the limit as $N \to \infty$, we see that $\overline{\tau}_n(\mu, \sigma_n) \to 0$ and hence asymptotically we have that $\tilde{\lambda}_n(\mu)$ converges to $\lambda_n(\mu)$. Moreover, we can develop an asymptotic error estimator. From Proposition 2.2, we have

$$\overline{\tau}_n(\mu, \tilde{\lambda}_n(\mu)) \approx \overline{\tau}_n(\mu, \lambda_n(\mu)) + (\tilde{\lambda}_n(\mu) - \lambda_n(\mu))\frac{\partial\overline{\tau}_n(\mu, \lambda_n(\mu))}{\partial\sigma}$$
$$= \frac{\lambda_n(\mu) - \tilde{\lambda}_n(\mu)}{\lambda_n(\mu)}. \tag{43}$$

Combining (42) and (43) gives the following asymptotic (relative) error estimator

$$\frac{|\lambda_n(\mu) - \tilde{\lambda}_n(\mu)|}{\lambda_n(\mu)} \lesssim \widetilde{\Delta}(\mu, \sigma_n). \tag{44}$$

---

[1]by computing the minimum eigenvalue corresponding to (38).

4. Port reduction

4.1. Empirical mode construction

In practice, for the basis functions of the port space $Z_p^{\mathcal{N}}$, we use a simple Laplacian eigenmode decomposition, corresponding to the eigenfunctions $\zeta_{p,k}$ of the following eigenproblem

$$\int_{\Gamma_p} \nabla \zeta_{p,k} \cdot \nabla v = \Lambda_{p,k} \int_{\Gamma_p} \zeta_{p,k} v, \quad \forall v \in Z_p^{\mathcal{N}}, \quad 1 \leq k \leq \mathcal{N}_p^{\Gamma}. \tag{45}$$

We can truncate the Laplacian eigenmode expansion in order to reduce $\mathcal{N}_p^{\Gamma}$ – often without any significant loss in accuracy of the method. However, we can obtain better results by tailoring the port basis functions to a specific class of problems. A strategy for the construction of such *empirical port modes* is presented in [10]. We briefly describe this strategy here and refer the reader to [10] for further detail.

A key observation is that, in a system of components, the solution on any given interior global port is "only" influenced by the parameter dependence of the two components that share this port *and* the solution on the non-shared ports of these two components. We shall exploit this observation to explore the solution manifold associated with a given port through a *pairwise training* algorithm.

Algorithm 1 Pairwise training (two components connected at global port Γ_p)

$S_{\text{pair}} = \emptyset$.

for $n = 1, \ldots, N_{\text{samples}}$ **do**

    Assign random parameters $\sigma \in [0, \sigma_{\text{max}}]$ and $\mu_i \in \mathcal{D}_i$ to component $i = 1, 2$

    (note the value of σ is the same for both components).

    On all non-shared ports, assign random boundary conditions.

    Solve $\mathcal{B}(u(\mu, \sigma), v; \mu; \sigma) = 0, \quad \forall v \in X_{\mathcal{S}}(\mu, \sigma)$

    Extract solution $u|_{\Gamma_p}$ on shared port.

    Subtract the average and add to snapshot set:

$$S_{\text{pair}} \leftarrow S \cup \left(u|_{\Gamma_p} - \frac{1}{|\Gamma_p|} \int_{\Gamma_p} u|_{\Gamma_p} \right).$$

end for

To construct the empirical modes we first identify groups of local ports on the components which may interconnect; the port spaces for all ports in each such group must be identical. For each pair of local ports within each group (connected to form a global port Γ_p), we execute Algorithm (1): we sample this $I = 2$ component system many (N_{samples}) times for random (typically uniformly or log-uniformly distributed) parameters over the parameter domain and for random boundary conditions on non-shared ports. For each sample we extract the solution on the shared port Γ_p; we then subtract its average and add the resulting zero-mean function to a snapshot set S_{pair}. Note that by construction all functions in S_{pair} are thus orthogonal to the constant function.

Upon completion of Algorithm 1 for all possible component connectivity within a library, we form a larger snapshot set S_{group} which is the union of all the snapshot sets S_{pair} generated for each pair. We

then perform a data compression step: we invoke proper orthogonal decomposition (POD) [15] (with respect to the $L^2(\Gamma_p)$ inner product). The output from the POD procedure is a set of mutually $L^2(\Gamma_p)$-orthonormal empirical modes that have the additional property that they are orthogonal to the constant mode.

4.2. Port-reduced system

In practice we use SCRBE – RB approximations for the bubble functions – but as we will see in the result section, the error introduced by RB approximation is very small and negligible compared to the error due to port reduction. As a consequence, we describe the port reduction procedure starting from the "truth" static condensation system (30), but we will in practice apply the port reduction to the SCRBE system (34). We recall that on port p the full port space is given as

$$Z_p^{\mathcal{N}} = \{\zeta_{p,1}, \cdots, \zeta_{p,\mathcal{N}_p^{\Gamma}}\}. \tag{46}$$

For each port, we shall choose a desired port space dimension $n_{\mathrm{A},p}$ such that $1 \leq n_{\mathrm{A},p} \leq \mathcal{N}_p^{\Gamma}$. We shall then consider the basis functions ζ_k, $1 \leq k \leq n_{\mathrm{A},p}$, as the *active* port modes (hence subscript $_{\mathrm{A}}$); we consider the $n_{\mathrm{I},p} = \mathcal{N}_p^{\Gamma} - n_{\mathrm{A},p}$ remaining basis functions ζ_k, $n_{\mathrm{A},p} + 1 \leq k \leq \mathcal{N}_p^{\Gamma}$, as *inactive* (hence subscript $_{\mathrm{I}}$). Note that $\mathrm{span}\{\zeta_{p,1}, \ldots, \zeta_{p,n_{\mathrm{A},p}}\} \subseteq Z_p^{\mathcal{N}}$. We then introduce

$$n_{\mathrm{A}} \equiv \sum_{p=1}^{n^{\Gamma}} n_{\mathrm{A},p}^{\Gamma}, \qquad n_{\mathrm{I}} \equiv \sum_{p=1}^{n^{\Gamma}} n_{\mathrm{I},p}^{\Gamma}, \tag{47}$$

as the number of total active and inactive port modes, respectively; and $n_{\mathrm{SC}} = n_{\mathrm{A}} + n_{\mathrm{I}}$ is the total number of port modes in the non-reduced system.

Next, we assume a particular ordering of the degrees of freedom in (30): we first order the degrees of freedom corresponding to the n_{A} active system port modes and then by the degrees of freedom corresponding to the n_{I} inactive system port modes. We may then interpret (30) as

$$\begin{bmatrix} \mathbb{B}_{\mathrm{AA}}(\mu,\sigma) & \mathbb{B}_{\mathrm{AI}}(\mu,\sigma) \\ \mathbb{B}_{\mathrm{IA}}(\mu,\sigma) & \mathbb{B}_{\mathrm{II}}(\mu,\sigma) \end{bmatrix} \mathbb{V}(\mu,\sigma) = \overline{\tau}(\mu,\sigma) \begin{bmatrix} \mathbb{A}_{\mathrm{AA}}(\mu,\sigma) & \mathbb{A}_{\mathrm{AI}}(\mu,\sigma) \\ \mathbb{A}_{\mathrm{IA}}(\mu,\sigma) & \mathbb{A}_{\mathrm{II}}(\mu,\sigma) \end{bmatrix} \mathbb{V}(\mu,\sigma), \tag{48}$$

where the four blocks in the matrices correspond to the various couplings between active and inactive modes; note that $\mathbb{B}_{\mathrm{AA}}(\mu) \in \mathbb{R}^{n_{\mathrm{A}} \times n_{\mathrm{A}}}$ and that $\mathbb{B}_{\mathrm{II}}(\mu) \in \mathbb{R}^{n_{\mathrm{I}} \times n_{\mathrm{I}}}$. Our port-reduced approximation $\widehat{\overline{\tau}}(\mu,\sigma)$ shall be given as the solution to the $n_{\mathrm{A}} \times n_{\mathrm{A}}$ system

$$\mathbb{B}_{\mathrm{AA}}(\mu,\sigma)\mathbb{V}_{\mathrm{A}}(\mu,\sigma) = \widehat{\overline{\tau}}(\mu,\sigma)\mathbb{A}_{\mathrm{AA}}(\mu,\sigma)\mathbb{V}_{\mathrm{A}}(\mu,\sigma),$$

$$\mathbb{V}_{\mathrm{A}}(\mu,\sigma)^T \mathbb{A}_{\mathrm{AA}}(\mu,\sigma)\mathbb{V}_{\mathrm{A}}(\mu,\sigma) = 1 \tag{49}$$

in which we may discard the (presumably large) $\mathbb{B}_{\mathrm{II}}(\mu,\sigma)$ and $\mathbb{A}_{\mathrm{II}}(\mu,\sigma)$ blocks; however the $\mathbb{B}_{\mathrm{IA}}(\mu,\sigma)$-block is required later for residual evaluation in the context of *a posteriori* error estimation.

4.3. Port reduction error estimator

We put a $\widehat{\cdot}$ on top of all the port reduced quantities. Suppose we have found σ_n such that $\widehat{\overline{\tau}}_n(\mu, \sigma_n) = 0$ with eigenvector of size n_{SC} in the non-reduced space

$$\widehat{\mathbb{V}}_n(\mu, \sigma_n) = \left[\begin{array}{c} \mathbb{V}_{\mathrm{A},n}(\mu, \sigma_n) \\ 0 \end{array} \right].$$

We can expand $\widehat{\mathbb{V}}_n(\mu, \sigma_n)$ in terms of the eigenvectors $\mathbb{V}_m(\mu, \sigma_n)$ of the non reduced space

$$\widehat{\mathbb{V}}_n(\mu, \sigma_n) = \sum_{m=1}^{n_{\mathrm{SC}}} \alpha_m(\mu, \sigma_n) \mathbb{V}_m(\mu, \sigma_n).$$

Since $\widehat{\overline{\tau}}_n(\mu, \sigma_n) = 0$, we can reasonably assume that $|\overline{\tau}_n(\mu, \sigma_n)| = \min_{1 \le m \le n_{\mathrm{SC}}} |\overline{\tau}_m(\mu, \sigma_n)|$. We now look at the following residual

$$\mathbb{B}(\mu, \sigma_n) \widehat{\mathbb{V}}_n(\mu, \sigma_n) = \sum_{m=1}^{n_{\mathrm{SC}}} \alpha_m(\mu, \sigma_n) \mathbb{B}(\mu, \sigma_n) \mathbb{V}_m(\mu, \sigma_n)$$

$$= \sum_{m=1}^{n_{\mathrm{SC}}} \alpha_m(\mu, \sigma_n) \overline{\tau}_m(\mu, \sigma_n) \mathbb{A}(\mu, \sigma_n) \mathbb{V}_m(\mu, \sigma_n)$$

so using the $\mathbb{A}(\mu, \sigma_n)$ orthogonality of the $\mathbb{V}_m(\mu, \sigma_n)$ we obtain

$$\|\mathbb{B}(\mu, \sigma_n) \widehat{\mathbb{V}}_n(\mu, \sigma_n)\|_{\mathbb{A}(\mu,\sigma_n)^{-1}}^2 = \sum_{m=1}^{n_{\mathrm{SC}}} |\overline{\tau}_m(\mu, \sigma_n)|^2 \|\alpha_m(\mu, \sigma_n) \mathbb{A}(\mu, \sigma_n) \mathbb{V}_m(\mu, \sigma_n)\|_{\mathbb{A}(\mu,\sigma_n)^{-1}}^2$$

$$\ge |\overline{\tau}_n(\mu, \sigma_n)|^2 \sum_{m=1}^{n_{\mathrm{SC}}} \|\alpha_m(\mu, \sigma_n) \mathbb{A}(\mu, \sigma_n) \mathbb{V}_m(\mu, \sigma_n)\|_{\mathbb{A}(\mu,\sigma_n)^{-1}}^2$$

$$= |\overline{\tau}_n(\mu, \sigma_n)|^2 \| \sum_{m=1}^{n_{\mathrm{SC}}} \alpha_m(\mu, \sigma_n) \mathbb{A}(\mu, \sigma_n) \mathbb{V}_m(\mu, \sigma_n)\|_{\mathbb{A}(\mu,\sigma_n)^{-1}}^2$$

$$= |\overline{\tau}_n(\mu, \sigma_n)|^2.$$

where we use the Euclidean norm derived from the $\mathbb{A}(\mu, \sigma_n)^{-1}$ scalar product. We thus obtain the following error bound

$$\widehat{\Delta}(\mu, \sigma_n) \equiv \|\mathbb{B}(\mu, \sigma_n) \widehat{\mathbb{V}}_n(\mu, \sigma_n)\|_{\mathbb{A}(\mu,\sigma_n)^{-1}} \ge |\overline{\tau}_n(\mu, \sigma_n)|.$$

Finally, we recover an error estimator for the eigenvalue $\lambda_n(\mu)$ of the original eigenproblem. Assuming $\widehat{\lambda}_n(\mu)$ is close to $\lambda_n(\mu)$, we can then use Proposition 2.2 as in (43), and we get the relative error estimator

$$\frac{|\lambda_n(\mu) - \widehat{\lambda}_n(\mu)|}{\lambda_n(\mu)} \lesssim \widehat{\Delta}(\mu, \sigma_n).$$

It is important to note that $\widehat{\Delta}(\mu, \sigma_n)$ will only decrease linearly in the residual, whereas the actual eigenvalue error is expected to decrease quadratically in the residual. This is due to the fact that port reduction can be viewed as a Galerkin approximation over a subspace of the skeleton space $X_{\mathcal{S}}(\mu, \sigma)$, and in that framework several *a priori* and *a posteriori* error results demonstrate the quadratic convergence of the eigenvalue [16]. As a consequence the effectivity of the error estimator $\widehat{\Delta}(\mu, \sigma_n)$ is expected to decrease with $n_{\mathrm{A},p}$.

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Note that

$$\mathbb{B}(\mu, \sigma_n)\widehat{\mathbb{V}}_n(\mu, \sigma_n) = \begin{bmatrix} 0 \\ \mathbb{B}_{\mathrm{IA}}(\mu, \sigma_n)\mathbb{V}_{\mathrm{A},n}(\mu, \sigma_n) \end{bmatrix},$$

and so the computation of the residual requires the additional assembly of $\mathbb{B}_{\mathrm{IA}}(\mu, \sigma_n)$, which does not generate an important extra computation since in practice we will consider $n_{\mathrm{A}} \ll n_{\mathrm{I}}$. On the contrary, the computation of the norm $\| \cdot \|_{\mathbb{A}(\mu, \sigma_n)^{-1}}$ requires the assembly and inversion of $\mathbb{A}(\mu, \sigma_n)$, the full Schur complement stiffness matrix, which would potentially eliminate any speed-up obtained by the port reduction. This computational issue is resolved by using an upper bound for $\|\cdot\|_{\mathbb{A}(\mu, \sigma_n)^{-1}}$ which is based on a non-conforming version $\mathbb{A}'(\mu, \sigma_n)$ of the stiffness operator and a parameter independent preconditioner: the former permits online computation of small matrice inverses locally on each component, and the latter allows us to precompute non-reduced matrices and their Cholesky decompositions in an offline stage. The entire procedure is described in detail in [10].

5. Numerical results

5.1. Linear elasticity

We consider linear elasticity in a non-dimensional form: we nondimensionalize space with respect to a length d_0 which will correspond to the beam width in the following, we nondimensionalize the Young's modulus E with respect to a reference value E_0, and we nondimensionalize time with respect to $\sqrt{\frac{\rho d_0}{E_0}}$, where ρ is the mass density. The non dimensional linear elasticity free vibration equation then reads

$$-\mathcal{A}U = \frac{\partial^2 U}{\partial t^2}, \tag{50}$$

where $\mathcal{A}$ is a linear second order differential operator in space and $U(x, t)$ is the displacement vector. Assuming that the free vibration solution is of the form $U(x, t) = u(x)cos(\omega t)$, the problem is equivalent to solving the eigenproblem

$$\mathcal{A}u = \omega^2 u. \tag{51}$$

In variational form, the operator $\mathcal{A}$ corresponds to the bilinear form [13]

$$a(w, v; \mu) \equiv \int_\Omega C_{ijkl}(\mu)\epsilon_{ij}(w)\epsilon_{kl}(v) \tag{52}$$

where we assume summation on repeated indices; $a(\cdot, \cdot)$ is defined on the space of admissible displacements $V = \{v = (v_1, v_2, v_3) | v_i \in H^1(\Omega); v_i = 0 \text{ on } \Gamma_0 \subset \partial\Omega\}$, and $\epsilon_{ij}(v) = \frac{1}{2}(\partial_i v_j + \partial_j v_i)$. We will consider isotropic materials, in which case the coefficents $C_{ijkl}(\mu)$ are functions of only two parameters: Poisson's ratio ν and Young's modulus E. In the following we always fix $\nu = 0.3$, and allow E to vary, hence E is part of the vector of parameters μ. More precisely, the parametric dependence reads

$$C_{ijkl}(\mu) = \frac{E\nu}{(1+\nu)(1-2\nu)}\delta_{ij}\delta_{kl} + \frac{E}{2(1+\nu)}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

We also define the mass bilinear form

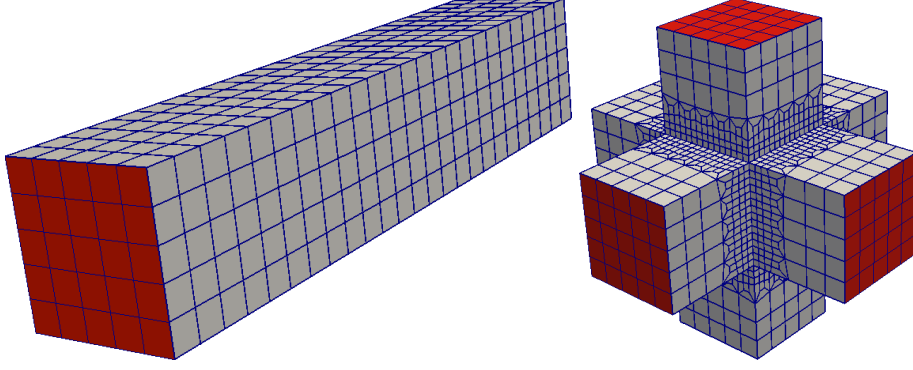$$m(w, v; \mu) \equiv \int_\Omega w_i v_i. \tag{53}$$

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Figure 1: The component library: a beam (left) and a connector (right). Components can connect at square ports shown in red. All ports have the same shape and same discretization.

The eigenproblem in variational form finally reads: find $\lambda(\mu) \in \mathbb{R}_{>0}$ and $u(\mu) \in V$ such that

$$a(u(\mu), v; \mu) \quad = \quad \lambda(\mu)m(u(\mu), v; \mu), \quad \forall v \in V, \tag{54}$$

$$m(u(\mu), u(\mu); \mu) \quad = \quad 1. \tag{55}$$

Note that $\lambda(\mu) = \omega^2(\mu)$ – the eigenvalue is the frequency squared.

5.2. Component library

We consider a linear elasticity library of two components shown in Figure 1: a beam and a connector. The FE hexahedral meshes are shown in Figure 1, and in all the following we use first order approximation with trilinear elements. The components can connect at square ports of dimension 1×1 with $\mathcal{N}_p^\Gamma = 3 \times 36 = 108$ degrees of freedom. The beam has two parameters: the Young's modulus $E \in [0.5, 2]$ and the length scaling $s \in [0.5, 2]$, where the beam is of length $5s$. The connector has one parameter, the Young's modulus $E \in [0.5, 2]$. Finally, for the shift parameter σ, we consider the range $[0, 0.01]$, based on the fact that the local minimum eigenvalues of the two components are larger than 0.01 for the previous E and s parameter ranges. For each component, we build RB bubble spaces of size $N = 10$ using a Greedy algorithm [17], for the parameter ranges previously defined. We also perform a pairwise training for the component pair beam-connector to build empirical port modes as described in Section 4.1; and we build a parameter independent preconditioner (necessary for the computation of $\widehat{\Delta}$) using parameter values $E = 0.5$ and $s = 0.5$.

5.3. Simple beam

We first present a simple example where we compare with beam theory to demonstrate that the FE resolution is adequate and that we capture the different modes. We consider a clamped-clamped uniform beam of square section, with thickness $d = 1$ and length $L = 40$, and Young's modulus $E = 1$. Table 1 presents the first eight eigenvalues obtained by different methods: Euler Bernoulli model [18], Timoshenko model [18], global FEM and SCRBE with and without port reduction (in which the beam is constructed as the concatenation of eight beam components). The eigenvalues (which we recall are

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Figure 2: The fourth (pure bending) and fifth (pure torsion) beam eigenmodes.

the frequencies squared) are quite small as the beam is of large aspect ratio. The SCRBE results are obtained by connecting eight beam components together with length parameter $s = 1$, using RB spaces of size $N = 10$; no port reduction corresponds to $n_{\mathrm{A,p}} = \mathcal{N}_p^\Gamma = 108$, and for port reduction we use $n_{\mathrm{A,p}} = 20$ active port modes. The global FEM results are obtained using a global mesh corresponding to eight beam component meshes stitched together, hence SCRBE and FEM are based on the same mesh and FE resolution.

We observe that the beam models do not capture some eigenvalues; these correspond to torsional modes that are not taken into account in Euler Bernoulli and Timoshenko models which consider only bending displacement. Note that for a beam with a square section, the bending and torsion is decoupled and the eigenmodes are either pure bending or pure torsion (see Figure 2). For the modes that are pure bending (λ_1, λ_2, λ_3, λ_4, λ_6, λ_7), we observe a good agreement between all methods. Note that it is well known that Euler Bernoulli is better for long wavelength and/or slender beams; Timoshenko is better for shorter wavelength and/or shorter beams. Not surprisingly, the FE (and SCRBE) eigenvalues are closer to Euler Bernoulli for lower modes and closer to Timoshenko for higher modes. The SCRBE (with or without port reduction) and global FEM give results that have an actual relative difference less than 10^{-4}. For the SCRBE without port reduction, we also give the relative error estimate $\widetilde{\Delta}$ in Table 1, which corresponds to the relative error between the SCRBE and the "truth" static condensation: it is at most 10^{-6}, which confirms that the error introduced by RB is negligible. For the SCRBE with port reduction and $n_{\mathrm{A,p}} = 20$, the relative error estimate is $\widehat{\Delta}$ and corresponds to the relative error between SCRBE with and without port reduction: it is about 10^{-2}, which overstimates the actual relative error, but nonetheless indicates a very good agreement between SCRBE eigenvalues with and without port reduction. We also observe that the SCRBE does capture all the torsional modes. Note finally that the SCRBE eigenvalues are obtained using a root finding algorithm: in practice we set the tolerance[2] to 10^{-10} as this is a couple orders of magnitude smaller than the RB relative error estimator $\widetilde{\Delta}$, thus making the root finding error negligible with respect to RB error (and also port reduction error).

5.4. Bridge structure

We are now ready to consider larger systems with more complicated connections which will better exercise the RB and port reduction capabilities. Towards this end, we consider a system of 30 components, corresponding to a bridge structure, shown in Figure 3 with its first three eigenmodes; in that case we choose $E = 0.5$ and $s = 1$ for all components. In the following, we will provide systematic analysis of

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[2]the tolerance applies to $\widetilde{\overline{\tau}}_n(\mu, \sigma) = 0$, hence it corresponds to a relative tolerance for $\widetilde{\lambda}_n(\mu)$.

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
Euler Bernoulli	1.6294e-05	1.2381e-04	4.7583e-04	1.3003e-03	—	2.9016e-03	5.6603e-03	—
Timoshenko	1.6204e-05	1.2224e-04	4.6524e-04	1.2560e-03	—	2.7622e-03	5.2991e-03	—
Global FEM	1.6612e-05	1.2489e-04	4.7327e-04	1.2708e-03	2.0732e-03	2.7775e-03	5.2916e-03	6.1912e-03
SCRBE $n_{\mathrm{A,p}} = 108$	1.6612e-05	1.2489e-04	4.7327e-04	1.2708e-03	2.0732e-03	2.7775e-03	5.2916e-03	6.1912e-03
SCRBE $n_{\mathrm{A,p}} = 20$	1.6612e-05	1.2489e-04	4.7327e-04	1.2708e-03	2.0732e-03	2.7775e-03	5.2916e-03	6.1912e-03
$\widetilde{\Delta}$	1.4418e-06	2.0695e-07	7.9612e-08	9.6913e-08	5.4576e-09	4.1418e-07	1.0262e-06	8.8249e-09
$\widehat{\Delta}$	5.5488e-03	7.3845e-03	8.4207e-03	7.4811e-03	3.3180e-02	8.3262e-03	8.9995e-03	4.7761e-03

Table 1: Eigenvalues for a clamped-clamped uniform beam of square section, with thickness $d = 1$ and length $L = 40$. The estimators $\widetilde{\Delta}$ and $\widehat{\Delta}$ correspond relative errors.
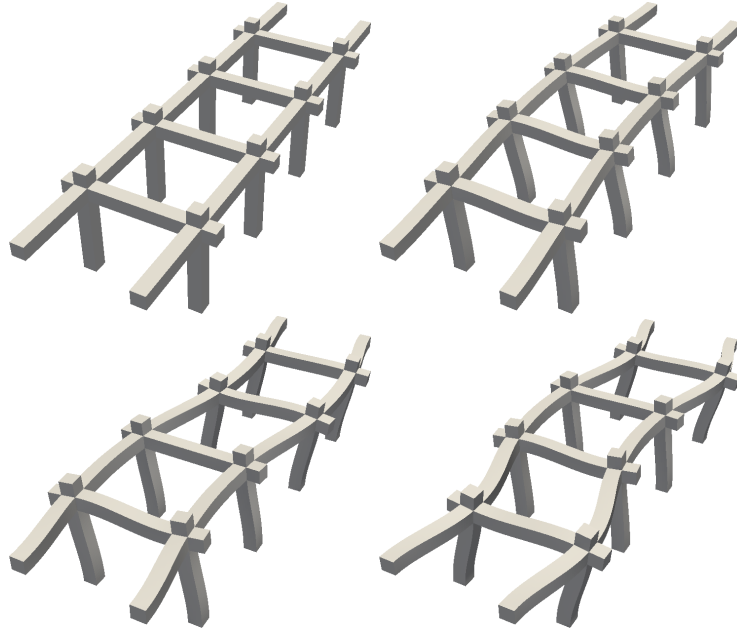

Figure 3: A bridge structure: all of the "open ports" of beam components are clamped. The first three displacement eigenmodes are shown.
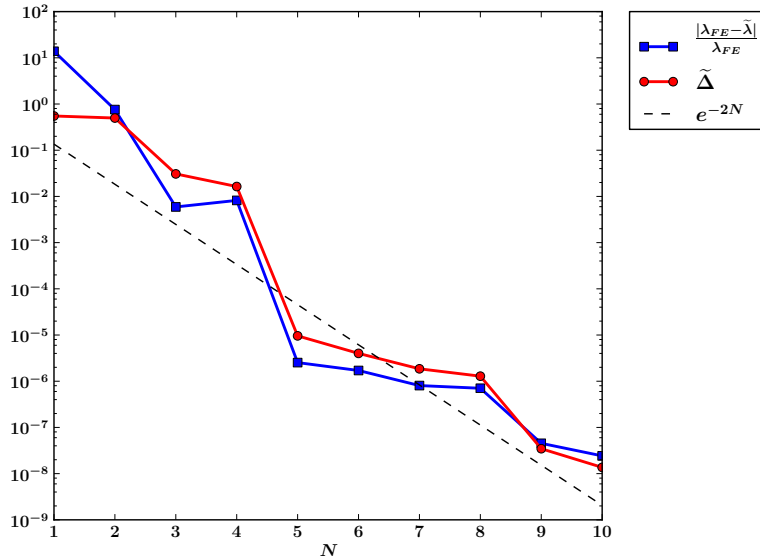
Figure 4: First eigenvalue convergence with respect to the size N of RB spaces.

the RB and port reduction convergence and also performance of the *a posteriori* error estimates; finally, we will report detailed timings to confirm the computational advantage for large problems.

We first show in figure 4 the convergence of the first eigenvalue with respect to the size N of the RB spaces used for bubble approximations. Note that we did not compute the eigenvalue with the "truth" static condensation, because it would be very computationally intensive, hence the reference value for λ is the value obtained with a global FEM, denoted λ_{FE}[3]. We observe that we obtain exponential convergence, hence we provide a significant improvement compared to standard CMS approaches. We also observe that the RB relative error estimator $\widetilde{\Delta}$ is accurate – it overestimates the actual error by at most one order of magnitude; moreover, for a sufficiently large N, the RB relative error estimator $\widetilde{\Delta}$ is very small, hence justifying the fact that we can neglect the error due to RB error approximation when introducing port reduction.

We now fix $N = 10$ and consider port reduction. We show in Figure 5 the convergence of the first eigenvalue with respect to the number of port modes, for both the regular Laplacian modes and the empirical modes: the advantage obtained with the empirical modes is obvious, and we also observe that $\widehat{\Delta}$ does not converge as fast as the true error, which is due to the fact that it is only a linear function of the norm of the residual, as explained in Section 4.3.

We now consider computational savings compared to a global FEM approach. We report in table 2 the first eigenvalue computation time and error bounds with and without port reduction; in table 3 the same results are reported for the tenth eigenvalue computation. Note that we use the Krylov-Schur method

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[3]The global FEM eigenvalue is not expected to be exactly the same as what would be obtained with FE static condensation – in theory they should be the same, but the different computational paths lead to different numerical results – hence it explains why $\frac{|\lambda_{FE}-\tilde{\lambda}|}{\lambda_{FE}}$ does not converge to zero, and why $\widetilde{\Delta}$ gets smaller than $\frac{|\lambda_{FE}-\tilde{\lambda}|}{\lambda_{FE}}$ for N big enough.
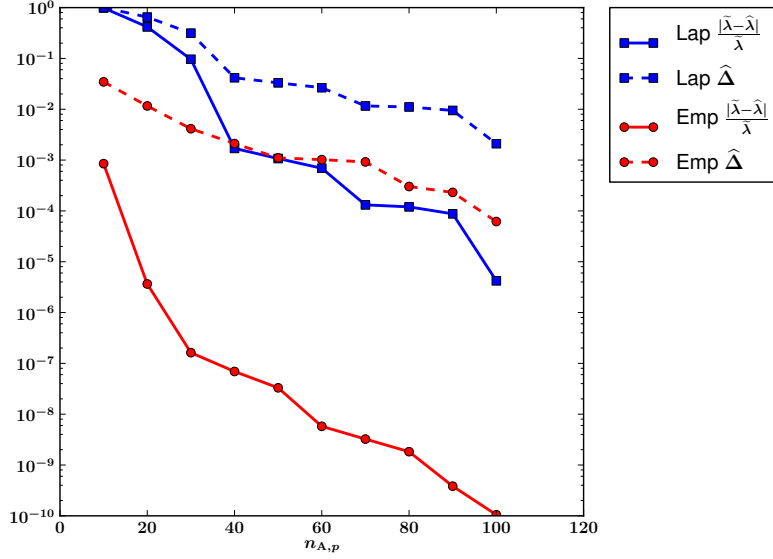
Figure 5: First eigenvalue convergence with respect to the number $n_{\mathrm{A,p}}$ of active port modes: blue corresponds to Laplacian eigenmodes ("Lap"), red corresponds to empirical modes ("Emp"). The reference eigenvalue $\widetilde{\lambda}$ is the one obtained for $n_{\mathrm{A,p}} = \mathcal{N}_p^{\Gamma} = 108$.

with the shift and invert transform, provided by the SLEPC library [19]. For SCRBE, we use a direct LU solver for matrix inversion (thanks to the moderate number of dofs); for FE we use an iterative solver: conjugate gradient with incomplete Cholesky preconditioner (1 cpu) or Jacobi preconditioner (16 cpu). For the first eigenvalue, without port reduction, the SCRBE yields a speed-up factor of 60 compared to FE with 1 cpu, and a speed-up factor of 11 compared to FE with 16 cpu. For the tenth eigenvalue, the decrease in computational time is even larger – 112 compared to FE with 1 cpu and 23 compared to FE with 16 cpu. Computing eigenvalues that are not at an extremity of the spectrum is a more difficult problem, and in that case the computational advantage of the SCRBE is even more obvious. We observe that the relative difference between the SCRBE and FE eigenvalues is less than 10^{-6}. We now add port reduction: we retain only some of the port (empirical) modes for the eigenproblem solution. When retaining about a fifth of the port modes ($n_{\mathrm{A,p}} = 20$), the speed-up factors for the first and tenth eigenvalues are 960 and 2500 compared to FE with 1 cpu, 180 and 520 compared to FE with 16 cpu. The SCRBE eigenvalues are again within a 10^{-6} relative distance from the FE eigenvalues, and the port reduction relative error estimator predicts a 1% relative error for the port-reduced eigenvalues – despite overestimation of the true error, this 1% relative error estimate is more than sufficient in an engineering context. Note that the SCRBE can be easily parallelized thanks to the component assembly description, but in this paper the SCRBE computations are done on a single CPU, hence the comparison of global FEM with 16 cpu to SCRBE with 1 cpu is strongly biased in favor of FEM.

For our formulation to remain well posed, we can only capture eigenvalues that lie in the σ range $[0, 0.01]$. Despite this limitation, we are able to compute up to $\lambda_{50} = 0.00994974$, which is more than

	number dofs	1st eigenvalue	$\widetilde{\Delta}(\mu,\sigma)$	$\widehat{\Delta}(\mu,\sigma)$	timing
Global FEM (1 cpu)	$\mathcal{N} = 267792$	0.000108903			32 min
Global FEM (16 cpu)	$\mathcal{N} = 267792$	0.000108903			6 min
SCRBE $n_{\mathrm{A,p}} = \mathcal{N}_p^{\Gamma} = 108$	$n_{\mathrm{A}} = n_{\mathrm{SC}} = 6480$	0.000108903	1.36e-08		32s
SCRBE $n_{\mathrm{A,p}} = 20$	$n_{\mathrm{A}} = 1200$	0.000108903		1.16%	2s

Table 2: First eigenvalue computation for a bridge assembled as 30 components (Figure 3). $E = 0.5$ and $s = 1$ everywhere.

	number dofs	10th eigenvalue	$\widetilde{\Delta}(\mu,\sigma)$	$\widehat{\Delta}(\mu,\sigma)$	timing
Global FEM (1 cpu)	$\mathcal{N} = 267792$	0.00194414			125 min
Global FEM (16 cpu)	$\mathcal{N} = 267792$	0.00194414			26 min
SCRBE $n_{\mathrm{A,p}} = \mathcal{N}_p^{\Gamma} = 108$	$n_{\mathrm{A}} = n_{\mathrm{SC}} = 6480$	0.00194414	2.50e-08		67s
SCRBE $n_{\mathrm{A,p}} = 20$	$n_{\mathrm{A}} = 1200$	0.00194415		1.67%	3s

Table 3: Tenth eigenvalue computation for a bridge assembled as 30 components (Figure 3). $E = 0.5$ and $s = 1$ everywhere.

enough in this structural analysis case where only the lowest frequency modes are likely to resonate.

We demonstrated that SCRBE has a computational advantage relative to FE, but the same decrease in computational time could in theory be obtained with CMS. One crucial advantage of SCRBE with respect to CMS (in addition to convergence rate) is its flexibility with respect to parameter variations. Thanks to the RB approximations at the component level, we can modify the component parameters and recompute the eigenproblem solution seamlessly. We show in Figure 6 the third eigenmode for different parameter variations: we can modify some of the beam lengths (Figure 6a), or we can make one half of the bridge stiffer than the other (Figure 6b).

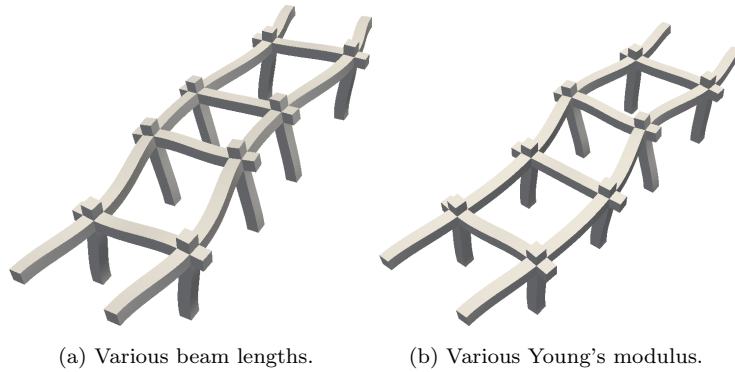(a) Various beam lengths.   (b) Various Young's modulus.

Figure 6: Comparison of the third eigenmode for different parameters. Left: for the middle beams, $s = 0.7$; for the beams adjacent to the middle beams, $s = 1.3$; for all other beams and all support beams, $s = 1$. Right: in the first half of the bridge $E = 2$, in the second half $E = 0.5$.
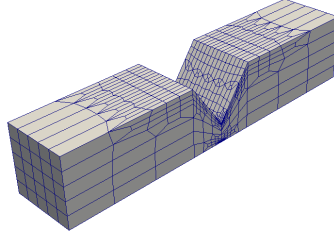
Figure 7: A notched beam component.

	Original system	Notched beam	Disconnected beam
λ_1	0.000108903	0.000109415	0.000107512
λ_2	0.000192330	0.000192394	0.000152107
λ_3	0.000351258	0.000351093	0.000336273
λ_4	0.000502429	0.000505191	0.000365281
λ_5	0.001227793	0.001206475	0.000874600

Table 4: Eigenvalue comparison when introducing defect in a beam.

Finally, one could argue that the examples presented in this section have relatively simple geometries and that the same results could be obtained by assembling components based on beam models, without solving the full linear elasticity equations. But one of the advantages of our approach (and the CMS in general) is its ability to handle arbitrary component shapes, allowing to tackle problems that are out of reach for beam and/or plate models. As an example, we introduce a notched beam component (Figure 7) in our original bridge system: consequently, a new mode appears between the ninth and tenth eigenmodes of the original system, as seen in Figure 8. This change of behaviour would be impossible to predict only using beam models. To emphasize the latter, we compare our notched beam case to a "disconnected beam" case (with zero-thickness crack, Figure 9c) that would be the only possible modeling of a defect with beam models: table 4 clearly shows that a notched beam introduce subtle changes in the spectrum whereas a disconnected beam completely modifies the spectrum.

Acknowledgements

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(a) 9th eigenmode of the original system

(b) 9th eigenmode with the notched beam

(c) 10th eigenmode of the original system

(d) 11th eigenmode with the notched beam

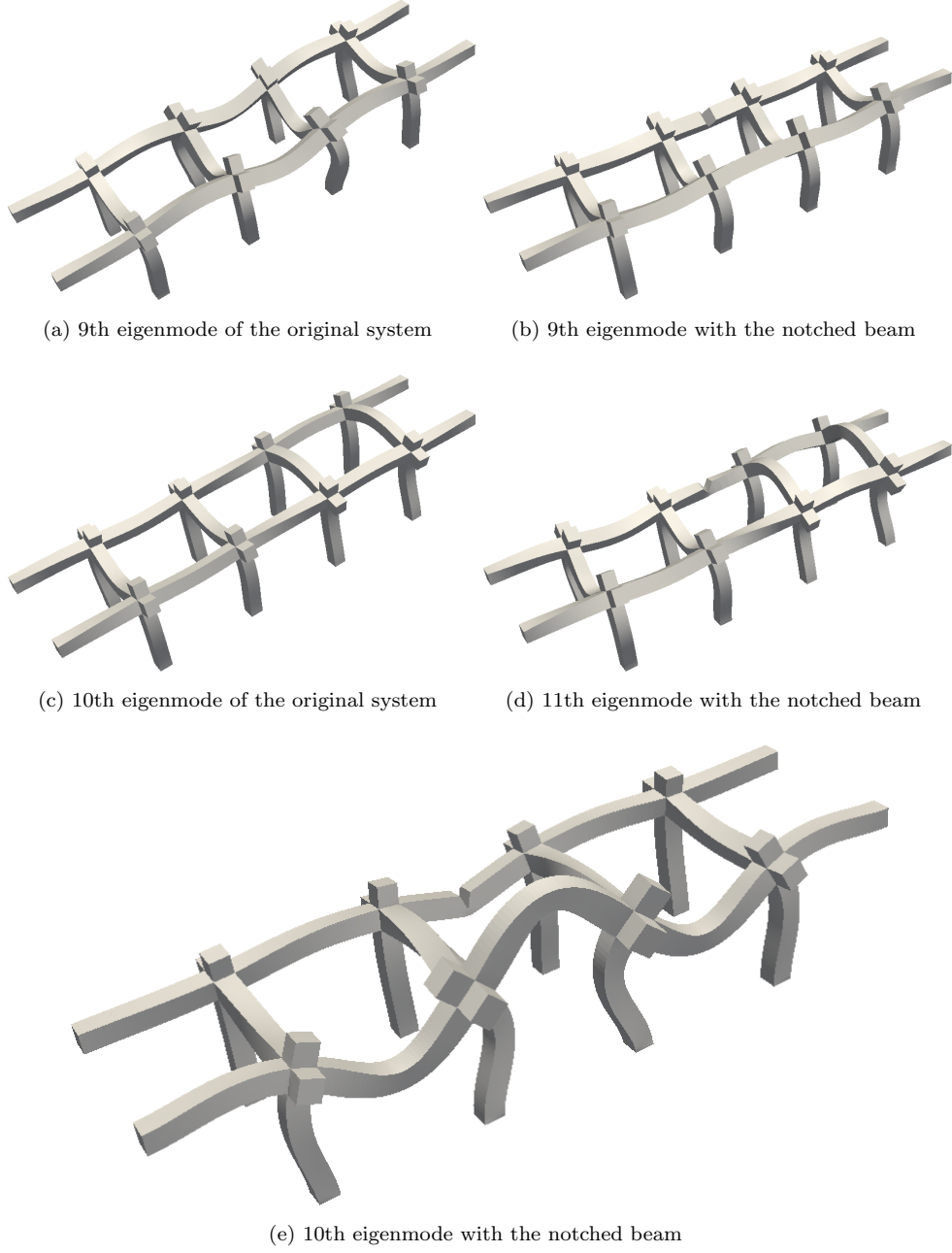(e) 10th eigenmode with the notched beam

Figure 8: Introducing a notched beam component in the bridge structure makes a new eigenmode appear in between the original 9th and 10th eigenmodes. For the original system, the eigenvalues are $\lambda_9 = 0.00182734$ and $\lambda_{10} = 0.00194414$. For the new system with a notched beam, the eigenvalues are $\lambda_9 = 0.00182379$, $\lambda_{10} = 0.00192261$ and $\lambda_{11} = 0.00193566$.
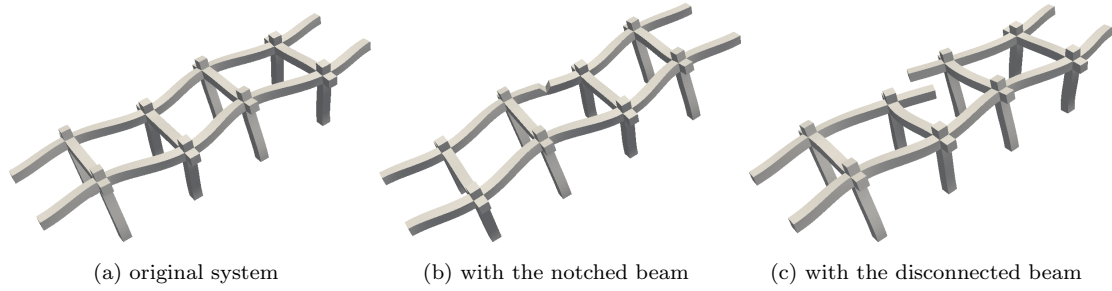
(a) original system   (b) with the notched beam   (c) with the disconnected beam

Figure 9: Comparison of the fourth eigenmode for different beam defects.

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